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ON THE DEFORMATION OF AN ELASTIC WEDGE PLATE REINFORCED BY A VARIABLE STIFFNESS BAR AND A METHOD OF SOLVING MIXED PROBLEMS

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The plane state of stress of an infinite elastic wedge reinforced by an infinite elastic bar along the bisectrix, whose stiffness varies as r^ω (r is the distance from the wedge apex), is considered. The problem is reduced to a first order difference equation for the displacement ω and is solved in closed form. The solution retains its meaning for $\omega = \pm\infty$, when the mentioned fundamental problem for the reinforced domain goes over into a mixed problem for the homogeneous domain. Therefore, the method proposed, which is applicable also to problems for rectangular, cylindrical and conical domains reinforced by bars, plates, circular slabs and shells of variable stiffness, is more general in specific respects than the Wiener-Hopf method.

Homogeneous [1—4] and inhomogeneous [5, 6] problems for an elastic wedge reinforced by constant stiffness bars have been studied earlier by using difference equations. Corresponding heat conduction and electromagnetic wave diffraction problems on a wedge have been solved in [7, 8], etc.

1. Let an elastic wedge-shaped plate $0 \leqslant r < \infty$, $-\alpha \leqslant \theta \leqslant \alpha$ of thickness h be welded completely along the bisectrix to an infinite elastic bar. The bar tensile and bending stiffnesses $2D_j(r)$ in the r, θ plane are expressed, respectively, by the equations

$$D_1(r) = \beta r + \gamma r^{1+\omega} \tag{1.1}$$

$$D_2(r) = \beta r^3 + \gamma r^{3+\omega} \tag{1.2}$$

where $\beta \geqslant 0$, $\gamma \geqslant 0$ and ω are any real numbers, where different numbers in (1.1) and (1.2) can be denoted by identical letters. The magnitudes of forces applied to the wedge or bar at the points $r = l_s$ will be denoted by the letters M, N, S with subscripts s, while the subscripts 0 and ∞ correspond to points of the wedge $r = 0$ and of the bar $r = \infty$ (see Fig. 1; the notation for the forces applied at the points $r = l_s$ and $r = \infty$

and shown by arrows is analogous to that mentioned for the point $r = 0$ and is omitted in the Figure).

Let us separate the problem posed into symmetric and skew-symmetric. Then the boundary conditions for the right half of the wedge decompose into three groups: general conditions

$$h\sigma_\theta (r, \alpha) = N_1\delta (r - l_1), \quad h\tau_{r\theta} (r, \alpha) = S_1\delta (r - l_1) \tag{1.3}$$

the contact conditions with the bar stretched without bending

$$v (r, 0) = 0 \tag{1.4}$$

$$\frac{\partial}{\partial r} D_1 (r) \frac{\partial}{\partial r} u (r, 0 + h\tau_{r\theta} (r, 0) = - S_3\delta (r - l_3) \tag{1.5}$$

$$h \int_0^\infty \tau_{r\theta} (r, 0) \, dr = S_2 - S_3 - S_\infty \tag{1.6}$$

and contact conditions with the bar bent without stretching

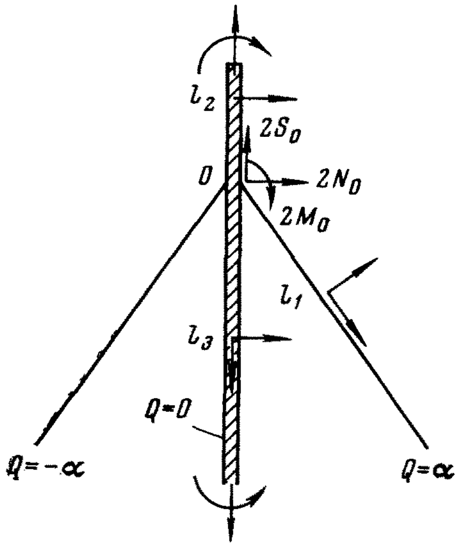

Fig. 1

$$u (r, 0) = 0 \tag{1.7}$$

$$\frac{\partial^2}{\partial r^2} D_2 (r) \frac{\partial^2}{\partial r^2} v (r, 0) - h\sigma_\theta (r, 0) = N_3\delta (r - l_3) \tag{1.8}$$

$$h \int_0^\infty \sigma_\theta (r, 0) \, dr = - N_2 - N_3 \tag{1.9}$$

$$h \int_0^\infty \sigma_\theta (r, 0) \, r dr = M_2 + N_2 l_2 - N_3 l_3 - M_\infty \tag{1.10}$$

Here $\delta (r)$ is the Dirac delta function.

2. Let us seek the solution of the symmetric problem as integrals [5] evaluated taking account of the boundary conditions (1.3) and (1.4)

$$2G \begin{Bmatrix} u (r, \theta) \\ v (r, \theta) \end{Bmatrix} = \frac{1}{2\pi i} \int_L \Big(\pm A (p) \Big[(p \pm \varkappa) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (p + 1) \theta \pm \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (p - 1) \theta - \Delta_1^+ \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (p - 1) \theta \Big] + B (p) \Big[(p \pm \varkappa) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (p + 1) \theta + \Delta_1^- \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (p - 1) \theta \pm \Delta_2^+ \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (p - 1) \theta \Big] \pm \frac{N_1 l_1^p}{hp} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} [(p - 1) (\alpha - \theta)] \mp \frac{S_1 l_1^p}{hp} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} [(p - 1) (\alpha + \theta)] \Big) \frac{dp}{r^p} \tag{2.1}$$

$$\Delta_1^{\pm} = \cos 2p\alpha \pm p\cos 2\alpha, \quad \Delta_2^{\pm} = \sin 2p\alpha \pm p\sin 2\alpha, \quad \varkappa = \frac{3-\nu}{1+\nu} \tag{2.2}$$

$$B(p) = (\Delta_2^{+})^{-1}\{A(p)(\Delta_1^{+} - p + \varkappa) - h^{-1}p^{-1}l_1^{p}[N_1 \sin(p-1)\alpha - S_1\cos(p-1)\alpha]\} \tag{2.3}$$

where ν is the Poisson's ratio, G is the shear modulus of the plate, and the contour L is the line $\operatorname{Re} p = \lambda$.

We substitute (2.1)—(2.3) into condition (1.5) and we group members with identical powers of r. In the first integral of the equality obtained

$$\frac{1}{2\pi i}\int_L \left\{\left[\frac{\beta p^2\Delta_3}{2G\Delta_2^{+}} - h(1+\varkappa)p\right]A(p) + \beta p q(p) - S_3 l_3^{p}\right\}\frac{dp}{r^{p+1}} + \frac{1}{2\pi i}\int_L \left[\frac{\gamma p(p-\omega)\Delta_3}{2G\Delta_2^{+}}A(p) + \gamma(p-\omega)q(p)\right]\frac{dp}{r^{p+1-\omega}} = 0 \tag{2.4}$$

$$q(p) = (2Gh\Delta_2^{+})^{-1}(p + \varkappa + \Delta_1^{-} + \Delta_2^{+})\, l_1^{p}\,[S_1\cos(p-1)\alpha - N_1\sin(p-1)\alpha]$$

$$\Delta_3 = 4\varkappa\sin^2 p\alpha + 4p^2\sin^2\alpha - (1+\varkappa)^2$$

Denoting the contents in the braces by $C(p)$ and replacing the argument p by $p-\omega$, we shift the contour L along the real axis by a quantity ω and denote it by L_1. Let us rewrite this equality in the form

$$\frac{1}{2\pi i}\int_{L_1} C(p-\omega)\frac{dp}{r^{p+1-\omega}} = \frac{1}{2\pi i}\int_L [F(p)C(p) + f(p)]\frac{dp}{r^{p+1-\omega}} \tag{2.5}$$

$$F(p) = \beta^{-1}\gamma p^{-1}(\omega - p)[1 - 2Gh(1+\varkappa)(\beta p\Delta_3)^{-1}\Delta_2^{+}]^{-1} \tag{2.6}$$

$$f(p) = F(p)[S_3 l_3^{p} - 2Gh(1+\varkappa)\Delta_2^{+}\Delta_3^{-1}q(p)] \tag{2.7}$$

Let us assume the function $C(p)$ to be (1) regular, and (2) to tend to zero as $|\operatorname{Im} p| \to \infty$ in a strip bounded by the lines $\operatorname{Re} p = \lambda$ and $\operatorname{Re} p = \lambda - \omega$. Then, according to the Cauchy theorem, the contour L_1 in (2.5) can be replaced by L and the problem reduces to a first order difference equation

$$C(p-\omega) = F(p)C(p) + f(p) \quad (p \in L) \tag{2.8}$$

The form of its solution and the position of the contour L are connected with the sign of the parameter ω, and the cases $\omega = 0$ and $\gamma = 0$ are perfectly elementary. The uniqueness of the solution of the problem (2.8) is established by using the analog of the Liouville theorem (see Theorem 7 in Sect. 101 of [9]) exactly as the uniqueness of the solution of the Riemann problem [10]. We shall always represent the coefficient of the problem (2.8) as the product $F(p) = F_1(p)F_2(p)$, where $F_1(p)$ is an elementary function; the function $F_2(p)$ satisfies the Hölder condition on L and has the index $\varkappa = 0$.

Let $\omega < 0$. Setting $\lambda < 0$, $S_0 = 0$, we consider different values of the parameters β and γ. For $\beta > 0$, $\gamma > 0$ we obtain

$$F_1(p) = \beta^{-1}\gamma p^{-1}(\omega - p) \qquad F_2^{-1}(p) = 1 - 2Gh(1+\varkappa)\beta^{-1}p^{-1}\Delta_2^{+}\Delta_3^{-1} \tag{2.9}$$

$$F_2(iy) = 1 + \varkappa_1 |y|^{-1} + O(e^{-2\alpha|y|}), \quad \varkappa_1 = Gh(1+\varkappa)\beta^{-1}\varkappa^{-1}$$

Let us write the canonical solution $C_0(p)$ of the homogeneous equation (2.8) by taking account of the conditions (1) and (2), as

$$C_0(p) = \pi\omega^{-1}p\,(\beta/\gamma)^{p/\omega}\sin^{-1}(\pi\omega^{-1}p)\,X(p) \tag{2.10}$$

where the function $X(p)$ is expressed in the Bantsuri form [3]

$$X(p) = F_2^{-1}(p)\,Y(p) \quad (\omega < \operatorname{Re} p \leqslant 0), \quad X(p) = Y(p) \quad (0 < \operatorname{Re} p \leqslant -\omega) \tag{2.11}$$

$$Y(p) = \exp\left\{-\frac{1}{2\omega i}\int_{-i\infty}^{i\infty}\operatorname{ctg}\frac{\pi(t-p)}{|\omega|}\ln F_2(t)\,dt\right\} \tag{2.12}$$

According to the asymptotics (2.9), the integral (2.12) exists only in the sense of the Cauchy principal value, and in this sense possesses exponential convergence at infinity.

By virtue of the same conditions (1), (2) and in connection with the presence of a simple pole at the point $p = 0$ for the function $f(p)$, the solution of the inhomogeneous equation (2.8) becomes [5]

$$C(p) = AC_0(p) + C_{10}(p)\,Z(p), \quad C_{10}(p) = C_0(p)\cos(\pi\omega^{-1}p) \tag{2.13}$$

$$Z(p) = W(p) - g(p) \quad (\omega < \operatorname{Re} p \leqslant 0), \quad Z(p) = W(p) \quad (0 < \operatorname{Re} p \leqslant -\omega) \tag{2.14}$$

$$W(p) = -\frac{1}{2\omega i}\int_{-i\infty}^{i\infty}\frac{g(t)\,dt}{\sin[\pi|\omega|^{-1}(t-p)]}, \quad g(p) = -\frac{f(p)}{C_{10}(p-\omega)} \tag{2.15}$$

where $C_{10}(p)$ is the solution of the homogeneous equation

$$C_1(p-\omega) = -F(p)\,C_1(p) \quad (p \in L)$$

We find the value of the constant A from the condition (1.6), whose left side is a transform of the function $\tau_{r\theta}(r, 0)$, which equals $(1+\varkappa)\,p\cdot A(p)$ at $p = 0$, and the unknown force S_∞ in the right side is determined by the equalities

$$S_\infty = \lim_{r\to\infty}\frac{1}{2\pi i}\int_L E(p)\left(\frac{\beta}{r^p} + \frac{\gamma}{r^{p-\omega}}\right)dp = -\beta\lim_{p\to 0} pE(p) \tag{2.16}$$

$$E(p) = (2G\Delta_2^+)^{-1}p\Delta_3 A(p) + q(p)$$

Here the contour integral has been replaced by a residue series at the poles to the right of the imaginary axis. Expression $A(p)$ in terms of $C(p)$ and passing to the limit as $p \to 0$ in (2.13), we obtain

$$-S_2\{[1 + 2Gh(1+\varkappa)^{-1}\beta^{-1}(2\alpha + \sin 2\alpha)]\,Y(-0)\}^{-1} \tag{2.17}$$

We investigate the nature of the contact stresses at the wedge apex. Closing the contour L in (2.1) on the left by a system of semicircles passing between the poles of the transform $\tau_{r\theta}(r, \theta)$, we obtain as $r \to 0$

$$\tau_{r\theta}(r, 0) = \operatorname{Res}\left[\frac{2A\pi G(1+\varkappa)(\beta/\gamma)^{p/\omega}\Delta_2^+ Y(p)}{\beta\omega\Delta_3\sin(\pi\omega^{-1}p)\,r^{p+1}}\right]_{p=\mu} \tag{2.18}$$

Here μ is the first pole of the function enclosed in the square brackets. It can be either the first zero $p = \omega$ of the function $\sin(\pi\omega^{-1}p)$ or the first zero $p = a_1$ of the function Δ_3 (there is a table of values of a_1 in [11]). Hence, if $\omega > -1$, then stresses $\tau_{r\theta}(r, 0) = O(r^{-\mu-1})$ growing without limit originate at the apex for any α, where μ is the closer point of $p = \omega$ and $p = a_1$ to the imaginary axis. If $\omega \leqslant -1$, then $\mu = a_1$, and the mentioned singularity holds only for $\alpha > \alpha^*$, where $\alpha^* = \arcsin(1+\nu)^{-1/2}$ [11]. When $r \to \infty$, then $\tau_{r\theta}(r, 0) = O(r^{-\mu-1})$; here μ is selected from the numbers ω and a_2, the first pole of the funcuon $F_2(p)$ for $\operatorname{Re} p > 0$.

Let $\beta = 0$ and $\gamma > 0$. Taking account of the change in (2.6), we obtain

$$F_2(p) = -\frac{\Delta_3}{2\varkappa\Delta_2^+}\operatorname{tg}\frac{\pi p}{2|\omega|}, \quad F_2(iy) = 1 + O(e^{-2\alpha|y|}) + O(e^{-\pi|\omega^{-1}y|}) \quad (2.19)$$

$$C_0(p) = \left[\frac{Gh(1+\varkappa)}{\gamma\varkappa|\omega|}\right]^{p/\omega} \frac{\Gamma(1-\omega^{-1}p)X(p)}{\cos(^1/_2\pi\omega^{-1}p)} \quad (2.20)$$

The rest of the solution (2.11) — (2.15) is conserved, and the contact stresses at the wedge apex retain their character. The bar stiffness is now zero at infinity, the whole principal vector of the external load is transmitted to the wedge. According to (2.16), $S_\infty = 0$, and we have in place of (2.17)

$$A = -S_2X^{-1}(0) - Z(0) \quad (2.21)$$

This essentially rational solution becomes inapplicable for $\omega = -\infty$. In order for the appropriate passage to the limit to be realizable, the more awkward formulas

$$F_1(p) = \frac{\gamma\varkappa(\omega - p)}{Gh(1+\varkappa)}\operatorname{ctg}\pi p, \quad F_2(p) = -\frac{\Delta_3\operatorname{tg}\pi p}{2\varkappa\Delta_2^+} \quad (2,22)$$

$$C_0(p) = \left[\frac{Gh(1+\varkappa)}{\gamma\varkappa}\right]^{p/\omega} \frac{\pi\omega^{-1}pT(p)X(p)}{\sin(\pi\omega^{-1}p)} \quad (2.23)$$

$$T(p) = \prod_{s=1}^{\infty}\Gamma\left(\frac{\pi s - {}^1/_2\pi + p}{-\omega}\right)\Gamma\left(1 - \frac{\pi s - p}{\omega}\right)\Gamma^{-1}\left(\frac{\pi s + p}{-\omega}\right)\Gamma^{-1} \times \left(1 - \frac{\pi s - {}^1/_2\pi - p}{\omega}\right)\left(1 - \frac{1}{2s}\right)^{2p/\omega+1}$$

should be used in place of (2.19) and (2.20). They yield the exponential convergence of the integral (2.12) and do not alter the expressions (2.21). A solution of the problem $C(p-\omega) = F_1(p)C(p)$ has been obtained bv the method of Bames who investigated (2.8) in 1904 [12]. The case $\beta > 0$, $\gamma = 0$ is equivalent to the elementary case $\omega = 0$ for any ω.

Let us construct the solution for $\omega > 0$. The bar stiffness in r now grows so that the displacements of the wedge and bar caused by the effect of the lumped forces S_0 and S_2 can be matched at the point $r = 0$, and the principal vector of the given load at infinity is transmitted entirely to the bar, therefore, $\lambda > 0$.

For $\beta > 0$, $\gamma > 0$, only (2.11) and (2.14) change in the solution (2.9) — (2.16)

$$X(p) = F_2^{-1}Y(p),\ Z(p) = W(p) - g(p) \quad (0 < \operatorname{Re} p \leqslant \omega) \quad (2.24)$$
$$X(p) = Y(p),\ Z(p) = W(p) \quad (-\omega < \operatorname{Re} p \leqslant 0)$$

From the equalities

$$S_\infty = -\gamma \lim_{p\to\omega} (p-\omega)E(p) = Y(-0)\,[A + W(-0)] \qquad (2.25)$$

and from the equilibrium condition for the wedge bar system

$$S_\infty + S_3 - S_2 - S_0 - N_1 \sin\alpha + S_1 \cos\alpha = 0 \qquad (2.26)$$

we obtain

$$A = (S_0 + S_2 - S_3 + N_1 \sin\alpha - S_1 \cos\alpha) Y^{-1}(-0) - W(-0) \qquad (2.27)$$

Let $\beta = 0,\ \gamma > 0$. Then the solution of the problem (2.8) which satisfies conditions (1) and (2) is determined by (2.13), (2.20), (2.24), (2.19), (2.12) and (2.15). In order to establish a connection between the quantities S_0 and S_2, let us use condition (1.6) according to which

$$S_2 = \lim_{p\to 0}\ [h\,(1+\varkappa)\,pA\,(p)] + S_3 + S_\infty = -C\,(0) + S_\infty$$

Computing $C\,(0)$ by means of (2.13), etc., and taking account of (2.25), we obtain

$$C\,(0) = S_\infty = Y(-0)\,[A + W(-0)] \qquad (2.28)$$

Therefore, $S_2 = 0$ for any A. The force applied to the reinforced wedge is transmitted completely to its apex, is the force S_0. Again (2.27) follows from (2.28) and (2.26). The equalities (2.22) and (2.23) must be taken instead of (2.19) and (2.20) in the solution which remains meaningful for $\omega = \infty$.

3. Let us consider the skew-symmetric problem. By virtue of (1.7), we have for the displacements (2.1)

$$A\,(p) = -(\Delta_2^-)^{-1}\ \{B\,(p)\,(\Delta_1^- + p + \varkappa) + h^{-1} p^{-1} l_1^p\,[N_1 \times \cos\,(p-1)\,\alpha - S_1 \sin\,(p-1)\,\alpha]\} \qquad (3.1)$$

Substituting (2.1) into (1.8), introducing the new unknown function

$$C\,(p) = B\,(p)\,[(2G\Delta_2^-)^{-1}\beta p^2\,(p^2-1)\Delta_3 + h\,(1+\varkappa)\,p] - \beta p^2\,(p^2-1)\,q\,(p) - N_3 l_3^p \qquad (3.2)$$

and assuming that it satisfies the previous conditions (1), (2) in the previous strip, we obtain (2.8) with the following values of the functions therein:

$$F\,(p) = \frac{\gamma\,(p-\omega)\,(p-\omega-1)}{\beta p\,(1-p)\,[1 + 2Gh\,(1+\varkappa)\,\beta^{-1} p^{-1}\,(p^2-1)^{-1}\Delta_2^-\Delta_3^{-1}]} \qquad (3.3)$$

$$f\,(p) = -F\,(p)\,[N_3 l_3^p + 2Gh\,(1+\varkappa)\,p\,\Delta_2^-\Delta_3^{-1} q\,(p)] \qquad (3.4)$$

$$q\,(p) = (2Ghp)^{-1} l_1^p\,\{[N_1 \cos\,(p-1)\,\alpha - S_1 \sin\,(p-1)\alpha] \times (\Delta_2^-)^{-1} + N_1 \sin\,(p-1)\alpha + S_1 \cos\,(p-1)\alpha\} \qquad (3.5)$$

Let $\omega < -1$. Then $\lambda < 0$, $M_0 = N_0 = 0$, and if $\beta > 0,\ \gamma > 0$, then

$$F_1\,(p) = -\beta^{-1}\gamma p^{-1}\,(p-1)^{-1}(p-\omega)(p-\omega-1) \qquad (3.6)$$
$$F_2\,(p) = F\,(p) F_1^{-1}\,(p)$$

The general solution of (2.8) is

$$C\,(p) = C_0\,(p)\{A_1 + A_2 \pi\omega^{-1}\,\mathrm{ctg}\,[\pi\omega^{-1}(p-1)] + \cos\,(\pi\omega^{-1} p)\,Z\,(p)\} \qquad (3.7)$$

$$C_0(p) = \pi\omega^{-1}p(p-1)(\beta/\gamma)^{p/\omega}\sin^{-1}(\pi\omega^{-1}p)X(p) \tag{3.8}$$

where the functions $X(p)$ and $Z(p)$ are determined by (2.11) — (2.15). Let us find the constants A_1 and A_2. Similarly to (2.16), we have

$$M_\infty = -\lim_{r\to\infty}\frac{1}{2\pi i}\int_L\left[\frac{B(p)\Delta_3}{2G\Delta_2^-} - q(p)\right](p+1)p(\beta r^{1-p}+\gamma r^{1+\omega-p})dp = \tag{3.9}$$
$$\lim_{p\to 1} p(1-p^2)[B(p)(2G\Delta_2^-)^{-1} - q(p)]$$

From this and from conditions (1.9) and (1.10), whose left sides equal $h(1+\varkappa)pB(p)$ for $p=0$ and $p=1$, there follows:

$$A_1 = \frac{N_2+N_3-(1+\varkappa)^{-1}(N_1\cos\alpha+S_1\sin\alpha)}{X(0)[t_2(0)-1]} + A_2\pi\omega^{-1}\operatorname{ctg}\pi\omega^{-1} \tag{3.10}$$

$$A_2 = (M_2+N_2l_2)\pi^{-1}\omega(\gamma/\beta)^{1/\omega}\sin\pi\omega^{-1}Y^{-1}(1) \tag{3.11}$$

The contact stresses $\sigma_\theta(r,0)$ at the wedge apex are determined by the residue of their transform $h(1+\varkappa)pB(p)$ at the first pole $p=\mu$ to the left of the imaginary axis. According to (2.1), (3.1), (3.2), (3.7) and (3.8), $\sigma_\theta(r,0) = O(r^{-\mu-1})$ as $r\to 0$. If $A_2=0$, then as in the symmetric problem, $\mu = a_1$, and a power singularity appears for $\alpha>\alpha^*$. If $A_2\neq 0$, then μ equals a_1 or $\omega+1$, and then as $\omega\to -1-0$ (d is a constant)

$$\sigma_\theta(r,0)\to dr^{-1+0} \tag{3.12}$$

Let us consider the case $\beta=0$, $\gamma>0$. We set

$$F_1(p) = \frac{\gamma\varkappa(p+1)(p-\omega)(p-\omega-1)}{Gh(1+\varkappa)\operatorname{tg}(^1/_2\pi|\omega|^{-1}p)} \tag{3.13}$$
$$F_2(p) = -\frac{\Delta_3}{2\varkappa\Delta_2^-}\operatorname{tg}\frac{\pi p}{2|\omega|}$$

We write the solution of (2.8) as

$$C(p) = C_0(p)[A_1\omega^{-1}\operatorname{ctg}(\pi\omega^{-1}p) + A_2 + \cos(\pi\omega^{-1}p)Z(p)] \tag{3.14}$$

$$C_0(p) = Q^{p/\omega}\Gamma\left(\frac{\omega-p}{\omega}\right)\Gamma\left(\frac{\omega+1-p}{\omega}\right)\Gamma\left(\frac{p+1}{|\omega|}\right)\sin\frac{\pi p}{2\omega}X(p) \tag{3.15}$$
$$Q = Gh(1+\varkappa)\varkappa^{-1}\gamma^{-1}|\omega|^{-3}$$

evaluating the functions $X(p)$ and $Z(p)$ by means of (2.11) — (2.15). Since $M_\infty=0$, we obtain from conditions (1.9) and (1.10)

$$A_1 = 2N_2\pi^{-1}\omega\sin\pi\omega^{-1}X^{-1}(0) \tag{3.16}$$

$$A_2 = \frac{(N_3l_3-N_2l_2-M_2)\omega Q^{-1/\omega}}{\Gamma(-\omega^{-1})\Gamma(-2\omega^{-1})\sin(^1/_2\pi\omega^{-1})X(1)} - \left[\frac{2N_2\omega}{\pi X(0)} + Z(1)\right]\cos\frac{\pi}{\omega} \tag{3.17}$$

The stresses $\sigma_\theta(r,0)$ at the wedge apex remain the same as for $\beta\neq 0$. For large $|\omega|$, it is necessary to set in (3.14)

$$F_1(p) = \frac{\gamma\varkappa(p+1)(p-\omega)(p-\omega-1)}{Gh(1+\varkappa)}\operatorname{ctg}\pi p, \quad F_2(p) = -\frac{\Delta_3\operatorname{tg}\pi p}{2\varkappa\Delta_2^-} \tag{3.18}$$

$$C_0(p) = \frac{\pi p}{2\omega}(Q|\omega|)^{p/\omega}\Gamma\left(\frac{\omega+1-p}{\omega}\right)\Gamma\left(\frac{p+1}{-\omega}\right)T(p)X(p) \tag{3.19}$$

$$A_2 = \frac{2(N_3 l_3 - N_2 l_2 - M_2)(Q\,|\,\omega\,|)^{1/\omega}\omega^2}{\pi\Gamma(-\omega^{-1})\Gamma(-2\omega^{-1})T(1)X(1)}$$

without altering the remaining functions and the constant A_1.

Let $\omega \in [-1, 0)$. Then the terms with coefficient A_2 in the solutions (3.7) and (3.14) do not satisfy condition (1) (as is already noticeable in (3.12)) since the functions $\operatorname{ctg}[\pi\omega^{-1}(p-1)]$ and $\Gamma(1+\omega^{-1}-\omega^{-1}p)$ have poles at the point $p = 1 + n\omega$ (n is the integer part of $|\,\omega\,|^{-1}$). If $A_2 = 0$, these solutions are extended to the interval ω under consideration, but $M_2 + N_2 l_2 = 0$ therein for $\beta > 0$ according to (3.11), for example. In order to avoid such a constraint we construct a new solution (2.1), (2.2), (3.1), (3.2) in addition to that under consideration, for the homogeneous problem with one constant $A_2{}^*$ in place of A_2, which differs from this latter by just the location of the contour L, the line $\operatorname{Re} p = \lambda^*$. Assuming $\lambda^* > 1 + \omega$, $\lambda_1{}^* > \lambda^*$, we write the solution of (2.8), (3.13) for $f(p) \equiv 0$ in the case $\beta \neq 0$ as

$$C(p) = A_2{}^*\pi\omega^{-1}p(p-1)(\beta/\gamma)^{p/\omega}\sin[\pi\omega^{-1}(p-1)]X^*(p) \quad (3.20)$$

$$X^*(p) = F_2{}^{-1}(p)Y^*(p) \quad (\omega + \lambda_1{}^* < \operatorname{Re} p \leqslant \lambda_1{}^*) \quad (3.21)$$

$$X^*(p) = Y^*(p) \quad (\lambda_1{}^* < \operatorname{Re} p \leqslant \lambda_1{}^* - \omega) \quad (3.22)$$

$$Y^*(p) = \exp\left\{-\frac{1}{2\omega i}\int_{\lambda_1{}^*-i\infty}^{\lambda_1{}^*+i\infty}\operatorname{ctg}\frac{\pi(t-p)}{|\,\omega\,|}\ln F_2(t)\,dt\right\} \quad (3.23)$$

It is easy to verify that it satisfies conditions (1) and (2) if there are no poles or zeros for the function $F_2(p)$ in the strip $0 < \operatorname{Re} p < \lambda_1{}^*$. If such there are and they alter the index of the function $F_2(p)$ as L is shifted by $\lambda_1{}^*$ from the imaginary axis, then it can again be made equal to zero by multiplying $F_2(p)$ by $\operatorname{tg}^k \pi p$. The solution for $\beta = 0$ is constructed in an analogous manner taking account of (3.15). If $\omega > -1$, then A_1 can be found by means of (3.10), if $\omega = -1$, then

$$A_1 = \frac{N_2 + N_3 - (1+\varkappa)^{-1}(N_1\cos\alpha + S_1\sin\alpha)}{X(0)[F_2(0)-1]} + \frac{A_2{}^* X^*(0)}{X(0)} \quad (3.24)$$

$$X^*(0) = Y^*(n\omega)\prod_{s=1}^{n} F_2^{-1}(s\,|\,\omega\,|)$$

In the whole range $-1 \leqslant \omega < 0$

$$A_2{}^* = (M_2 + N_2 l_2)(\gamma/\beta)^{1/\omega}[Y^*(1)]^{-1} \quad (3.25)$$

Let $\omega > 0$, $l_2 = 0$. As for $\omega \in [1, 0)$, we shall seek the solution as the sum of integrals (2.1) taken over the contours (a) $\operatorname{Re} p = \lambda > 0$ and (b) $\operatorname{Re} p = \lambda^* > 1$ and corresponding to the cases of (a) all forces except M_0, M_2 acting on the reinforced wedge and (b) only the moments M_0, M_2 acting. Therefore, $f(p) \equiv 0$ for problem (b) in (2.8).

If $\beta \neq 0$, then for all ω the solution of this equation is expressed for problem (a) by (3.7) with $A_2 = 0$, (3.8), (2.24), (2.12), (2.15), (3.6) (3.3) – (3.5) and by (3.20) for problem (b), where

$$X^*(p) = F_2{}^{-1}(p)Y^*(p) \quad (1 \leqslant \operatorname{Re} p \leqslant 1 + \omega) \quad (3.26)$$

$$X^*(p) = Y^*(p) \quad (1 - \omega \leqslant \operatorname{Re} p \leqslant 1)$$

Here it is assumed for simplicity that the function $F_2(p)$ defined by (3.6) and (3.3) has no poles and zeros in the strip $0 < \mathrm{Re}\, p \leqslant 1$; this is valid in every case for $\alpha \leqslant {}^1/_6\, \pi$. It is possible to set $\lambda_1^* = 1$ in (3.23) for $Y^*(p)$. The constant A_2^* is found from (3.25), A_1 from (3.10) for $\omega \neq 1$ and from (3.24) for $\omega = 1$, where $X^*(0) = F(1) X^*(1)$ must be understood as the analytic continuation of the function $X^*(p)$ at the point $p = 0$. Now the forces M_0, N_0, whose magnitudes depend on A_1, A_2^* and can be found by integrating the stresses $\sigma_r(r, \theta)$ with respect to θ as $r \to 0$, act on the wedge apex together with M_2, N_2.

If $\beta = 0$, then we have for the problem (a) for all ω

$$C(p) = C_0(p) \sin[\pi\omega^{-1}(p+1)][A_1 \sin^{-1}(\pi\omega^{-1}p) + Z(p)] \quad (3.27)$$

The functions $C_0(p)$, $X(p)$ and $Z(p)$ are determined by (3.15), (2.24), (2.12), (2.15), (3.13), (3.3) — (3.5), where in (2.15)

$$C_{10}(p) = C_0(p) \sin[\pi\omega^{-1}(p-1)]$$

For the problem (b) with $\omega \in (0, 1]$

$$C(p) = A_2^* C_0(p)\, \omega^{-1} \operatorname{ctg}(\pi\omega^{-1}p) \quad (3.28)$$

and with $\omega \in (1, \infty)$

$$C(p) = A_2^* C_0(p)\, \omega^{-1} \sin[\pi\omega^{-1}(p+1)] \sin^{-1}(\pi\omega^{-1}p) \quad (3.29)$$

Here the function $C_0(p)$ is determined by (3.15) in which $X(p)$ must be replaced by the function $X^*(p)$ evaluating it by means of (3.26), (3.23) and (3.13) for $\lambda^* = 1$.

In the case of large ω, it is expedient to use, in the problems (a) and (b), the factorization (3.18) and the solution (3.19) which have a limit as $\omega \to \infty$, in order to improve the convergence of the integrals (2.12) and (3.23).

4. We consider the problem (1.1) — (1.10) and its solution for $\omega = \pm\infty$. Let $\omega = -\infty$. Then, the bar becomes inextensible and inflexible in the interval $0 \leqslant r \leqslant 1$, and for $r > 1$ its stiffnesses are determined only by the members βr and βr^3. Therefore, the main problem for the reinforced domain goes over into a mixed problem for a homogeneous domain with the fundamental boundary conditions (1.3), (1.4), (1.7) and the mixed conditions resulting from (1.5) and (1.8)

$$(\partial / \partial r)[u(r, 0)] = 0 \quad (r \in [0, 1]) \quad (4.1)$$

$$\beta \frac{\partial}{\partial r} r \frac{\partial}{\partial r} u(r, 0) + h\tau_{r\theta}(r, 0) = -S_3 \delta(r - l_3) \quad (r \in (1, \infty)) \quad (4.2)$$

$$\partial^2 / \partial r^2 [v(r, 0)] = 0 \quad (r \in [0, 1]) \quad (4.3)$$

$$\beta \frac{\partial^2}{\partial r^2} r^3 \frac{\partial^2}{\partial r^2} v(r, 0) - h\sigma_\theta(r, 0) = N_3 \delta(r - l_3) \quad (r \in (1, \infty)) \quad (4.4)$$

In the case of symmetric loads, the function $C(p)$ has the form (2.13), (2.17). Passing to the limit as $\omega \to -\infty$ in (2.10) — (2.15) (the conditions of the theorem of passage to a limit under an integral sign are satisfied here), we obtain for $\beta \neq 0$

$$C(p) = C_0(p)[A + Z(p)], \quad C_0(p) = X(p) \quad (4.5)$$

$$X(p) = F_2^{-1}(p) Y(p), \quad Z(p) = W(p) - g(p) \quad (\mathrm{Re}\, p \leqslant 0) \quad (4.6)$$

$$X(p) = Y(p), \quad Z(p) = W(p) \quad (\mathrm{Re}\, p > 0) \quad (4.7)$$

$$Y(p) = \exp\left\{-\frac{\operatorname{sign}\omega}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln F_2(t)\,dt}{t-p}\right\},\quad W(p) = -\frac{\operatorname{sign}\omega}{2\pi i}\int_{-i\infty}^{i\infty}\frac{g(t)\,dt}{t-p} \quad (4.8)$$

$$g(p) = C_0^{-1}(p)[\varepsilon S_3 l_3^p - 2Gh(1+\varkappa)\Delta_2^+\Delta_3^{-1}q(p)] \quad (4.9)$$

where $F_2(p)$ is evaluated by means of (2.9), $\varepsilon = 1$ for $l_3 > 1$, $\varepsilon = 0$ for $l_3 < 1$.

For $\beta = 0$, i.e. in the problem of an absolutely rigid bar of unit length welded to a wedge, the function $F_2(p)$ must be evaluated by means of (2.22). From (2.23), we obtain

$$C_0(p) = \pi^{-1/2}\Gamma(1/2 + \pi^{-1}p)\Gamma^{-1}(1 + \pi^{-1}p)\,X(p) \quad (4.10)$$

while the other equalities (4.5) — (4.9) and (2.21) retain their form.

If $\beta \neq 0$ in the skew-symmetric problem, we obtain according to (3.7) — (3.11)

$$C(p) = C_0(p)[A_1 + A_2(p-1)^{-1} + Z(p)] \quad (4.11)$$

$$C_0(p) = (p-1)\,X(p)$$

$$A_1 = \frac{N_2 + N_3 - (1+\varkappa)^{-1}(N_1\cos\alpha + S_1\sin\alpha)}{X(0)[l_2(0) - 1]},\quad A_2 = \frac{M_2 + N_2 l_2}{Y(1)}$$

Here we have (4.6) — (4.8), (3.3), (3.6) for $Z(p)$ and $X(p)$, where

$$g(p) = -C_0^{-1}(p)[N_3 l_3^p + 2Gh(1+\varkappa)p\Delta_2^-\Delta_3^{-1}q(p)] \quad (4.12)$$

If $\beta = 0$, then we obtain by virtue of (3.14), (3.16), (3.19)

$$C(p) = C_0(p)[A_1\pi^{-1}p^{-1} + A_2 + Z(p)]$$

$$C_0(p) = -1/2\sqrt{\pi p}\,(p+1)^{-1}\Gamma(1/2 + \pi^{-1}p)\,\Gamma^{-1}(1 + \pi^{-1}p)\,X(p)$$

$$A_1 = \frac{2N_2}{X(0)},\quad A_2 = -\frac{4(M_2 + N_2 l_2 - N_3 l_3)\Gamma(1+\pi^{-1})}{\sqrt{\pi}\,\Gamma(1/2 + \pi^{-1})\,X(1)}$$

The functions $X(p)$ and $Z(p)$ are expressed as (4.6) — (4.8), (3.18) and (4.12).

In the version $\omega = \infty$ the bar becomes nondeformable for $r \geqslant 1$, has the stiffnesses βr and βr^3 for $r < 1$. The intervals $[0, 1]$ and $(1, \infty)$ must be interchanged in the mixed conditions (4.1) — (4.4) governing this problem, while the main conditions are retained. If $\beta \neq 0$ the solution of the symmetric problem is given by (4.5) — (4.9), (2.27), (2.9); while the intervals $\operatorname{Re} p \leqslant 0$ and $\operatorname{Re} p > 0$ in (4.6), (4.7) change places, $\varepsilon = 1$ for $l_3 < 1$ and $\varepsilon = 0$ for $l_3 \geqslant 1$. If $\beta = 0$, then in contrast to the preceding function, $C_0(p)$ and $F_2(p)$ have the form (2.23) and (2.22). The solution of the skew-symmetric problem is constructed in an analogous manner.

5. Mixed problems of the form (1.3), (1.4), (1.7), (4.1) — (4.4) are usually solved by the Wiener-Hopf method [13]. If the functional equation to which they reduce is considered as a Riemann problem [10] or as a linear conjugate problem [14] (see [15], say), then solutions agreeing exactly with the limit solutions in Sect. 4 can be obtained by the Gakhov formulas. To do this, it is just necessary to reduce the index of the coefficient to zero in place of the fractionally-linear [10, 14] and polynomial factors [15] (for $\beta = 0$, when it is not zero) by using the function $\operatorname{tg}\pi p$. In contrast to the function $(1 - p^2)^{1/2}$ [15], for elementary factorizability as the ratio between four gamma functions, it conserves the exponential convergence of the integrals (4.8), (2.22) and yields a solution in the most efficient form.

It is clear that the method elucidated above can be applied to any mixed problems of elasticity theory which are solved in closed form by the Wiener-Hopf method. However, two kinds of problems must here be differentiated.

Among the first type examined in Sect. 4, are mixed problems for strips, circular and wedge stamps, for cylindrical, conical and wedge absolutely rigid coverings making contact with appropriate elastic domains, and also equivalent problems of symmetric plane cracks. They and their solutions are the fundamental limit problems and solutions for a strip, wedge, cylinder and cone (in particular, a half-plane and half-space), completely reinforced by variable stiffness bars, plates and shells.

The second type combines the mixed problems for elastic domains, partially reinforced by elastic bars, beam slabs, shells of constant or linearly increasing thickness, and problems with rectilinear nonsymmetric, cylindrical and conical semi-infinite cracks. These problems can be reduced to difference equations formally by inserting the boundary conditions of a fictitious exponentially increasing "stiffness". But, what is of greater interest, they are all obtained in the limit as $\omega \to \pm\infty$, from the fundamental problems for the corresponding completely reinforced domains in which a Winkler layer with variable coefficients of the foundation $K(r)$ varying as r^ω is sandwiched between an elastic thin-walled reinforcing element and an elastic spatial domain, or for domains in which the Winkler layer fills the infinite crack.

In problems for conical and wedge domains when the initial elasticity theory equations are transformed according to Mellin, the bending stiffnesses of the reinforcing bars, plates and shells must be given in the form (1.2), the tensile, shear and torsional stiffnesses must be given in the form (1.1) in problems on the bending of wedge plates reinforced by bars, and the coefficients of the foundation of the Winkler layer are expressed by

$$K(r) = (\beta r + \gamma r^{1+\omega})^{-1}$$

The nature of the change in stiffness D along the longitudinal coordinate x should be independent of the kind of stiffness in problems for rectangular and cylindrical domains when using a two-sided Laplace transform, i.e. for both bending and tension

$$D(x) = \beta + \gamma e^{\omega x}$$

The foundation coefficients $K(x)$ are

$$K(x) = (\beta + \gamma e^{\omega x})^{-1}$$

Problems for domains with a variable stiffness Winkler layer, which reduce to difference equations solvable by quadratures, will be considered separately, and in a somewhat different formulation.

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AN ENERGY IDENTITY IN PHYSICALLY NONLINEAR ELASTICITY AND ERROR ESTIMATES OF THE PLATE EQUATIONS

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An identity generalizing the Prager-Synge relationship [1, 2] in linear elasticity is deduced for a certain class of nonlinear elasticity laws. It permits estimation of the energy norm of the difference between some statically admissible stress field σ and the true field σ°, as well as between some kinematically admissible displacement field $\mathbf{u}$ and the true field $\mathbf{u}^\circ$, in terms of the energy norm for the difference between the fields σ and $\sigma(\mathbf{u})$ ($\sigma(\mathbf{u})$ is the stress field generated by the field $\mathbf{u}$). By using this identity, under definite constraints, it is proved that the root-mean-square value (over the volume of a plate) of the error in the solution of the plate equations derived from the volume problem by means of the Kirchhoff hypothesis, does not exceed $ch^{1/2}$, where c is a constant and h is the relative thickness. The Prager-Synge relationship [1, 2] was used in [3, 4] to estimate the error in linear shell theory. The results are related to [1 — 7].